Inverse medium scattering problems with Kalman filter techniques

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Let $k > 0$ be the wave number (fixed), and let $\theta \in \mathbb{S}^1$ be incident direction. We denote the incident field $u^{inc}(\cdot\,;\theta)$ with the direction θ by the plane wave of the form

$$u^{inc}(x;\theta) := \mathrm{e}^{ikx\cdot\theta},\ x \in \mathbb{R}^2.$$

Let Q be a bounded domain and let its exterior $\mathbb{R}^2 \setminus \overline{Q}$ be connected. We assume that $q \in L^\infty(\mathbb{R}^2)$, which refers to the inhomogeneous medium, satisfies $\mathrm{Im} q \geq 0$, and its support supp q is embed into Q, that is supp $q \Subset Q$. Let the scattered field u^{sca} be the solution of the following equations

$$\Delta u^{sca} + k^2(1+q)u^{sca} = -k^2 q u^{inc}(\cdot\,;\theta) \text{ in } \mathbb{R}^2,$$

$$\lim_{|x|\to\infty} \sqrt{|x|}\left(\frac{\partial u^{sca}}{\partial |x|} - iku^{sca}\right) = 0.$$

It is well known that there exists an unique solution $u^{sca} = u_q^{sca}(\cdot\,;\theta)$ of the above scattering problem, and it has the following asymptotic behaviour

$$u_q^{sca}(x;\theta) = \frac{\mathrm{e}^{ik|x|}}{\sqrt{|x|}}\Big\{u_q^\infty(\hat{x};\theta) + O\big(1/|x|\big)\Big\},\ |x| \to \infty,\quad \hat{x} := \frac{x}{|x|}.$$

The function u_q^∞ is called the *far field pattern*, and it has the form

$$u_q^\infty(\hat{x};\theta) = \frac{k^2}{4\pi}\int_Q \mathrm{e}^{-ik\hat{x}\cdot y}\left(u^{inc}(y;\theta) + u_q^{sca}(y;\theta)\right) q(y)dy =: \mathcal{F}_\theta q(\hat{x}),$$

where the *far field mapping* $\mathcal{F}_\theta : L^2(Q) \to L^2(\mathbb{S}^1)$ is defined in the second equality for each incident direction $\theta \in \mathbb{S}^1$.

We consider the inverse scattering problem to reconstruct the function q from the far field pattern $u^\infty(\hat{x}, \theta_n)$ for all directions $\hat{x} \in \mathbb{S}^1$ and several directions $\{\theta_n\}_{n=1}^N \subset \mathbb{S}^1$ with some $N \in \mathbb{N}$. In this talk, we follow **Kalman filter** of the **linear estimation**, which updates the state and its covariance matrix by using the **sequential measurements**. Our problem is **nonlinear** (that is, the mapping $\mathcal{F}_\theta : L^2(Q) \to L^2(\mathbb{S}^1)$ is nonlinear), so we employ the **linearization** (Fréchet derivative of $\mathcal{F}_\theta$). We propose two schemes called **Kalman filter Levenberg–Marquardt** (which is equivalent to the Levenberg–Marquardt for full measurements) and **Extended Kalman Filter**, which are given by **different linearization ways**, respectively.

This is a joint work with Roland Potthast (Deutscher Wetterdienst, Germany).